Electrical Circuits (2)

Lecture 8

Transient Analysis Part(2)

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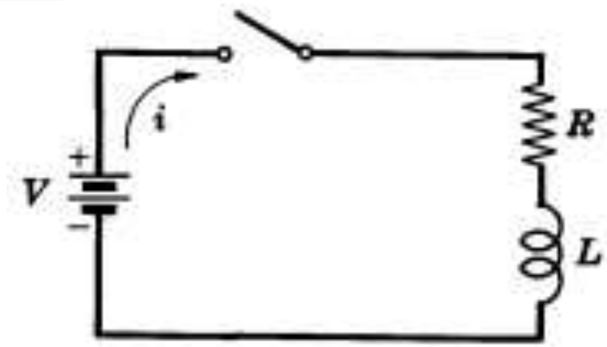
First-Order RL Transient Step-Response

- The switch "S" is closed at t = 0
- Apply KVL to the circuit in figure:



$$Ri + L\frac{di}{dt} = V$$

- Rearranging and using "D" operator notation :

$$\left(D + \frac{R}{L}\right)i = \frac{V}{L}$$

This Equation is a first order, linear differential equation

1. Complementary (Transient) Solution

The auxiliary equation is : $m + \frac{R}{L} = 0$

$$i = Ae^{mt} = Ae^{\frac{-R}{L}t}$$

$\tau = \frac{R}{L}$ Time constant

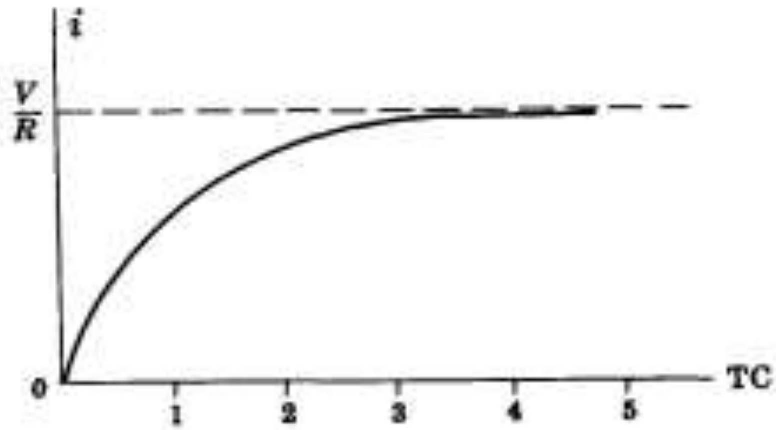
2. Particular (Steady-State) Solution

The steady-state value of the current for DC source is : $I_{ss} = \frac{V}{R}$

First-Order RL Transient Step-Response

- The total solution is:

$$i = Ae^{\frac{-R}{L}t} + \frac{V}{R}$$



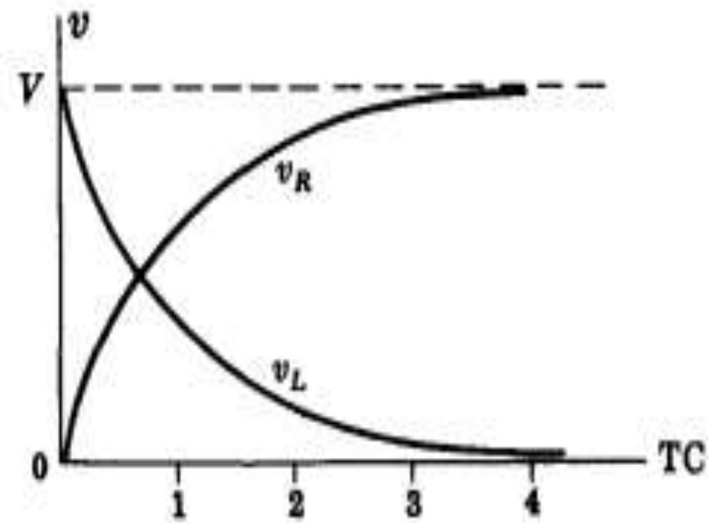
Since The initial current is zero:

$$0 = A + \frac{V}{R}$$

$$i = -\frac{V}{R}e^{-(R/L)t} + \frac{V}{R} = \frac{V}{R}(1 - e^{-(R/L)t})$$

- The voltage across the resistor is:

$$v_R = Ri = V(1 - e^{-(R/L)t})$$



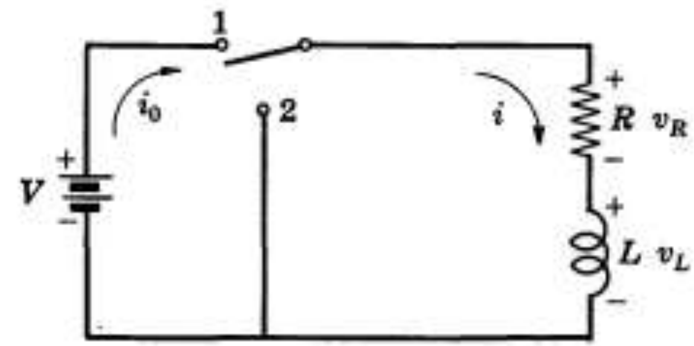
- The voltage across the inductor is:

$$v_L = L\frac{di}{dt} = L\frac{d}{dt}\left\{\frac{V}{R}(1 - e^{-(R/L)t})\right\} = Ve^{-(R/L)t}$$

$$v_R + v_L = V(1 - e^{-(R/L)t}) + Ve^{-(R/L)t} = V$$

First-Order RL Transient (Discharge)

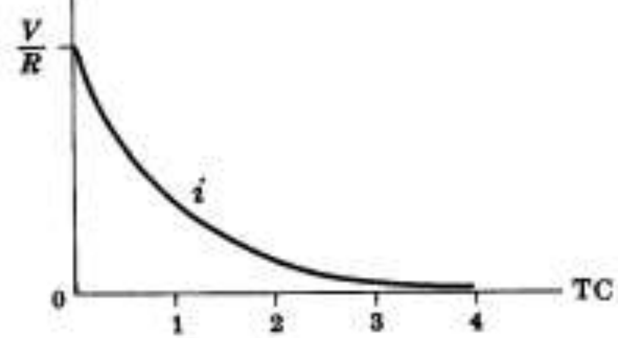
- The RL circuit shown in Figure contains an initial current of (V/R)
- The Switch "S" is moved to position"2" at t=0



$$L\frac{di}{dt} + Ri = 0 \qquad \text{or} \qquad \left(D + \frac{R}{L}\right)i = 0$$

- The solution is the transient (Complementary) part only.

$$i = ce^{-(R/L)t}$$



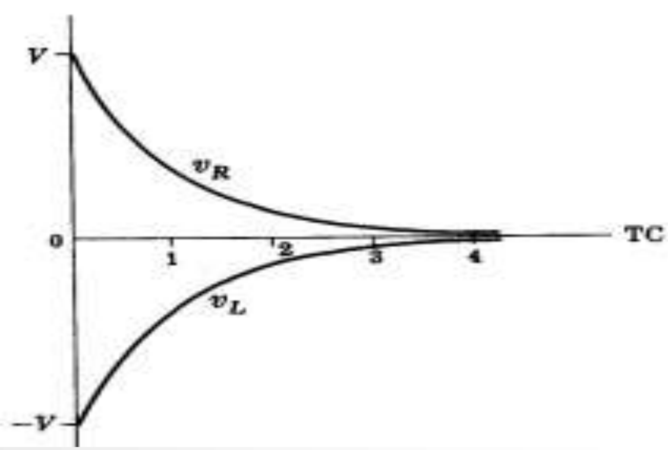
- Using the initial condition of the current, we get:

$$i = \frac{V}{R}e^{-(R/L)t}$$

- The corresponding voltages across the resistance and inductance are

$$v_R = Ri = Ve^{-(R/L)t}$$

$$v_L = L\frac{di}{dt} = -Ve^{-(R/L)t}$$



Examples

A series RL circuit with $R = 50$ ohms and $L = 10$ h has a constant voltage $V = 100$ v applied at $t = 0$ by the closing of a switch. Find (*a*) the equations for i, v_R and v_L, (*b*) the current at $t = .5$ seconds and (*c*) the time at which $v_R = v_L$.

(*a*) The differential equation for the given circuit is

$$50i + 10\frac{di}{dt} = 100 \quad \text{or} \quad (D+5)i = 10$$

the complete solution is $\quad i = i_c + i_p = ce^{-5t} + 2$

At $t = 0$, $i_0 = 0$ and $0 = c(1) + 2$ or $c = -2$. Then

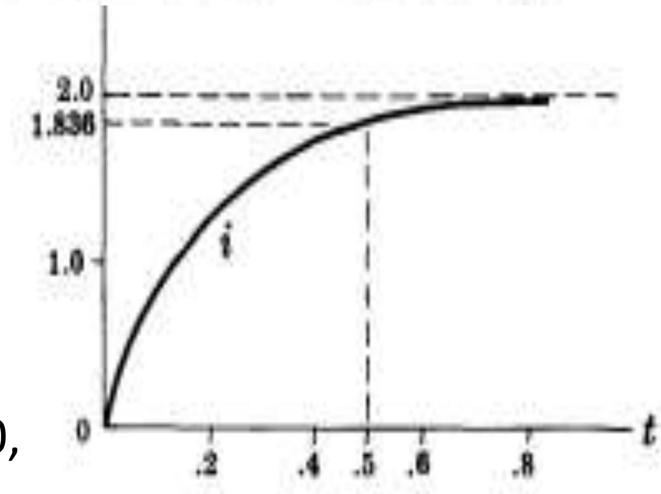
$$i = 2(1 - e^{-5t})$$

$$v_R = Ri = 100(1 - e^{-5t})$$

$$v_L = L\frac{di}{dt} = 100e^{-5t}$$

Examples

(b) Put $t = .5$ sec in (3) and obtain $i = 2(1 - e^{-5(.5)}) = 2(1 - .082) = 1.836$ amp.

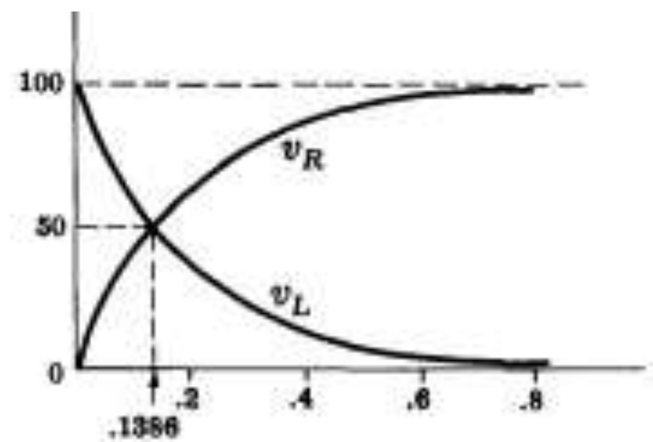


(b) For the two voltage to be equal:
each must be 50 volts since the applied voltage is 100,

$$v_L = 50 = 100e^{-5t}.$$

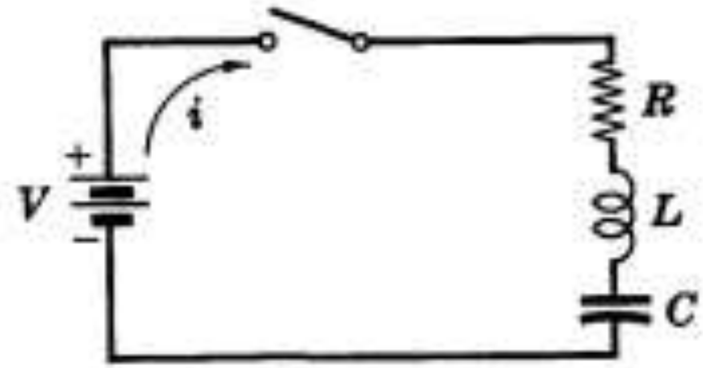
$$e^{-5t} = .5 \quad \text{or} \quad 5t = .693,$$

$$t = .1386 \text{ sec.}$$



Second-Order <u>RLC</u> Transient (Step Response)

- The Switch "S" is closed at t=0
- Applying KVL will produce the following Integro-Differential equation:



$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = V$$

- Differentiating, we obtain

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0 \quad \text{or} \quad \left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)i = 0$$

This second order, linear differential equation is of the homogeneous type with a particular solution of zero.

✓ The complementary function can be one of **<u>three different types</u> according to the roots of the auxiliary equation which** depends upon the relative magnitudes of R, L and C.

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0$$

Second-Order RLC Transient (Step Response)

We can Rewrite the auxiliary equation as:

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0$$

$$m^2 + 2\zeta\omega_0 m + \omega_0^2 = 0$$

$$\begin{cases} \frac{R}{L} = 2\zeta\omega_0 \\ \frac{1}{\sqrt{LC}} = \omega_0^2 \end{cases}$$

ζ : expontial dampling ratio

ω_0 : undamped natural frequency

- The roots of the equation (or natural frequencies):

$$\begin{cases} m_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ m_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$$

$$\begin{cases} m_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \\ m2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \end{cases}$$

$$\begin{cases} m_1 = -\sigma + \sqrt{\sigma^2 - \omega_0} \\ m_2 = -\sigma - \sqrt{\sigma^2 - \omega_0} \end{cases}$$

Second-Order RLC Transient (Step Response)

Case 1: Overdamped,

$\zeta > 1$ $\Rightarrow m_1, m_2$ are real and unequal

$$\frac{R}{2L} > \frac{1}{\sqrt{LC}}$$

$$\begin{cases} m_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ m_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$$

$$\sigma^2 > \omega_o^2$$

Natural response is the sum of two decaying exponentials:

$$i_{tr} = K_1 e^{m_1 t} + K_2 e^{m_2 t}$$

Case 2: Critically damped,

$\zeta = 1$ $\Rightarrow m_1, m_2$ are real and equal.

$$\frac{R}{2L} = \frac{1}{\sqrt{LC}}$$

$$\sigma^2 = \omega_o^2$$

$$m_1 = m_2 = -\omega_0$$

$$x_c(t) = e^{m_1 t}(B_1 + B_2 t)$$

Use the initial conditions to get the constants

Usually it is reduced to:

$$x_c(t) = B.t.e^{m_1 t}$$

Second-Order RLC Transient (Step Response)

Case 3: Underdamped,

$$\zeta < 1$$

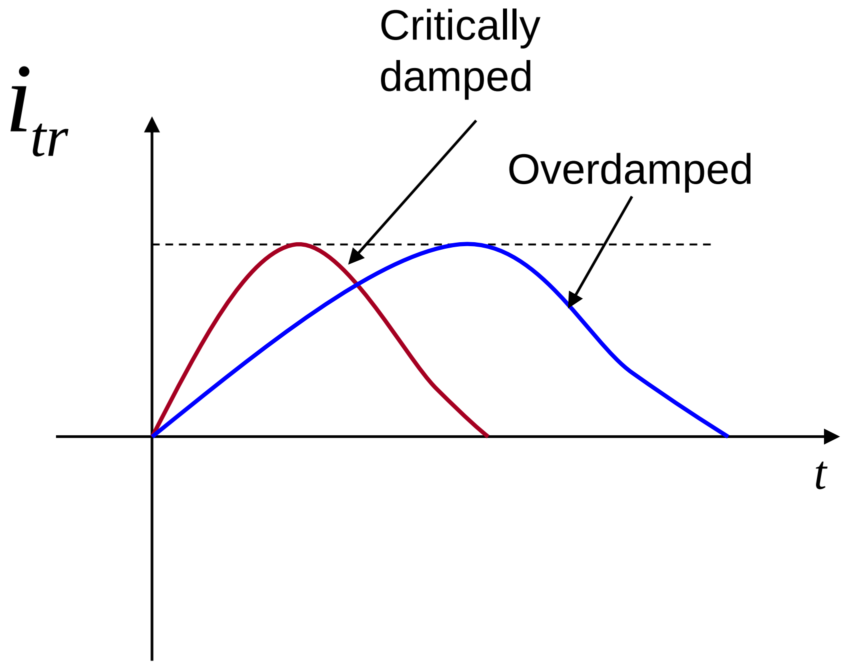
$$\frac{\mathrm{R}}{2\mathrm{L}} < \frac{1}{\sqrt{LC}} \qquad \Rightarrow m_1, m_2 \text{ are complex and conjugate.}$$

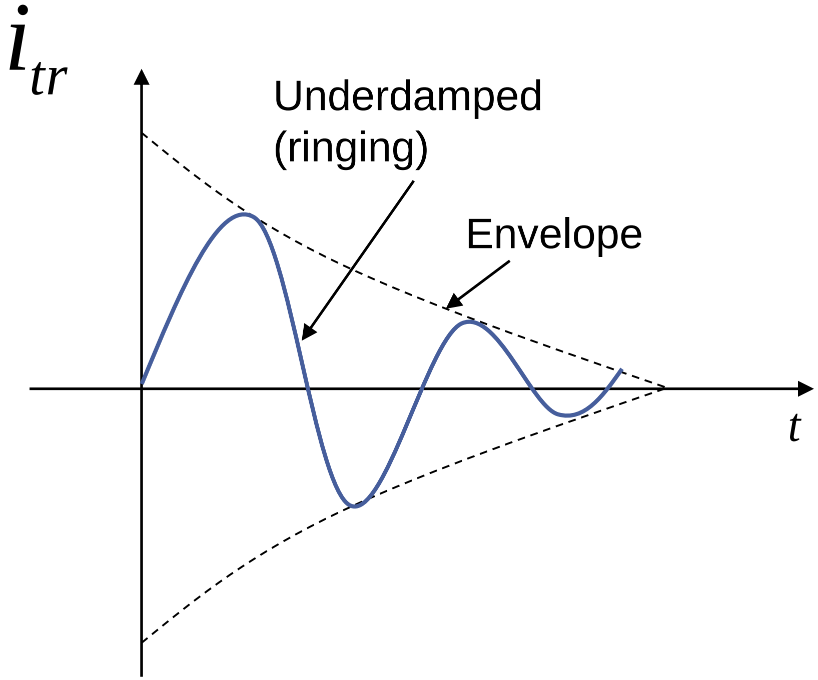
$$\sigma^2 < \omega_o^2$$

$$\begin{cases} m_1 = -\sigma + j\omega_d = (-\zeta\omega_0) + j(\omega_0\sqrt{1-\zeta^2}) \\ m_2 = -\sigma - j\omega_d = (-\zeta\omega_0) - j(\omega_0\sqrt{1-\zeta^2}) \end{cases}$$

Natural response is an exponentially damped oscillatory response:

$$i_{tr} = e^{-\sigma t}\{A_1 \cos(\omega_d\, t) + A_2 \sin(\omega_d\, t)\}$$





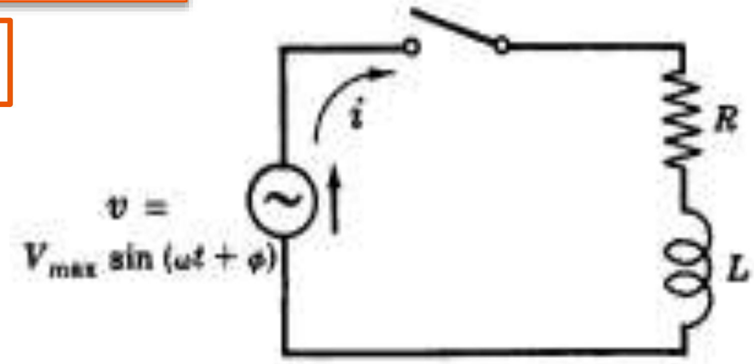
- ✓ The current in all cases contains the exponential decaying factor (damping factor) assuring that the final value is zero
- ✓ In other words, assuring that the complementary function decays in a relatively short time.

Alternating Current Transients

RL Sinusoidal Transient

$$Ri + L\frac{di}{dt} = V_{max}\sin(\omega t + \phi)$$

$$\left(D + \frac{R}{L}\right)i = \frac{V_{max}}{L}\sin(\omega t + \phi)$$



1. Complementary (Transient) Solution is the solution of the homogeneous 1st order DE

The same as before, The auxiliary equation is : $m + \frac{R}{L} = 0$

The complementary function is $i_c = ce^{-(R/L)t}$

2. Particular (Steady-State) Solution

The steady-state value of the current for ac source is :

$$I_{ss} = \frac{V_{max}}{\sqrt{X_L^2 + R^2}} Sin(wt + \phi - \tan^{-1}(\omega L / R))$$

Alternating Current Transients

RL Sinusoidal Transient

The complete solution is

$$i = i_c + i_p = ce^{-(R/L)t} + \frac{V_{\max}}{\sqrt{R^2+\omega^2L^2}}\sin\left(\omega t + \phi - \tan^{-1}\omega L/R\right)$$

Use the initial condition to find the value of c

$$i_0 = 0 = c(1) + \frac{V_{\max}}{\sqrt{R^2+\omega^2L^2}}\sin\left(\phi - \tan^{-1}\omega L/R\right)$$

$$c = \frac{-V_{\max}}{\sqrt{R^2+\omega^2L^2}}\sin\left(\phi - \tan^{-1}\omega L/R\right)$$

Substituting by the constant values, we get:

$$i = e^{-(R/L)t}\left[\frac{-V_{\max}}{\sqrt{R^2+\omega^2L^2}}\sin\left(\phi - \tan^{-1}\omega L/R\right)\right] + \frac{V_{\max}}{\sqrt{R^2+\omega^2L^2}}\sin\left(\omega t + \phi - \tan^{-1}\omega L/R\right)$$

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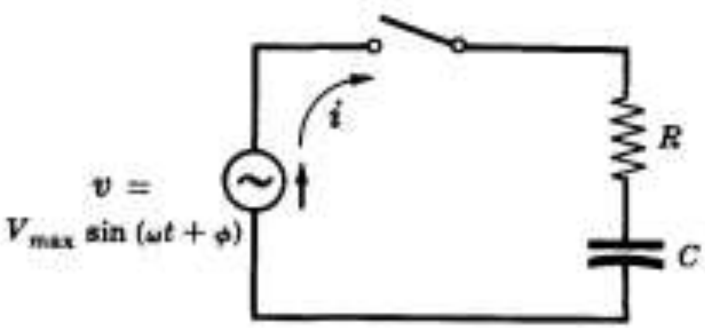
Alternating Current Transients

RC Sinusoidal Transient

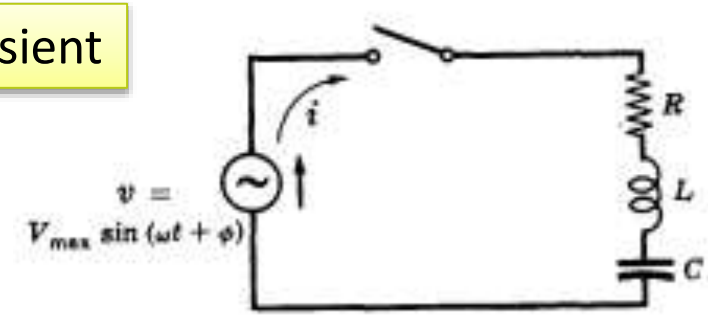


$$i = e^{-t/RC}\left[\frac{V_{max}}{R}\sin\phi - \frac{V_{max}}{\sqrt{R^2 + (1/\omega C)^2}}\sin(\phi + \tan^{-1} 1/\omega CR)\right]$$

$$+ \frac{V_{max}}{\sqrt{R^2 + (1/\omega C)^2}}\sin(\omega t + \phi + \tan^{-1} 1/\omega CR)$$

Alternating Current Transients

RLC Sinusoidal Transient



- Particular (Steady-State) Solution

$$i_p = \frac{V_{max}}{\sqrt{R^2 + (1/\omega C - \omega L)^2}} \sin\left(\omega t + \phi + \tan^{-1}\frac{(1/\omega C - \omega L)}{R}\right)$$

- Complementary(Transient) Solution

The complementary function is identical to that of the DC series RLC circuit examined previously where the result was overdamped, critically damped or oscillatory, depending upon R, L and C.

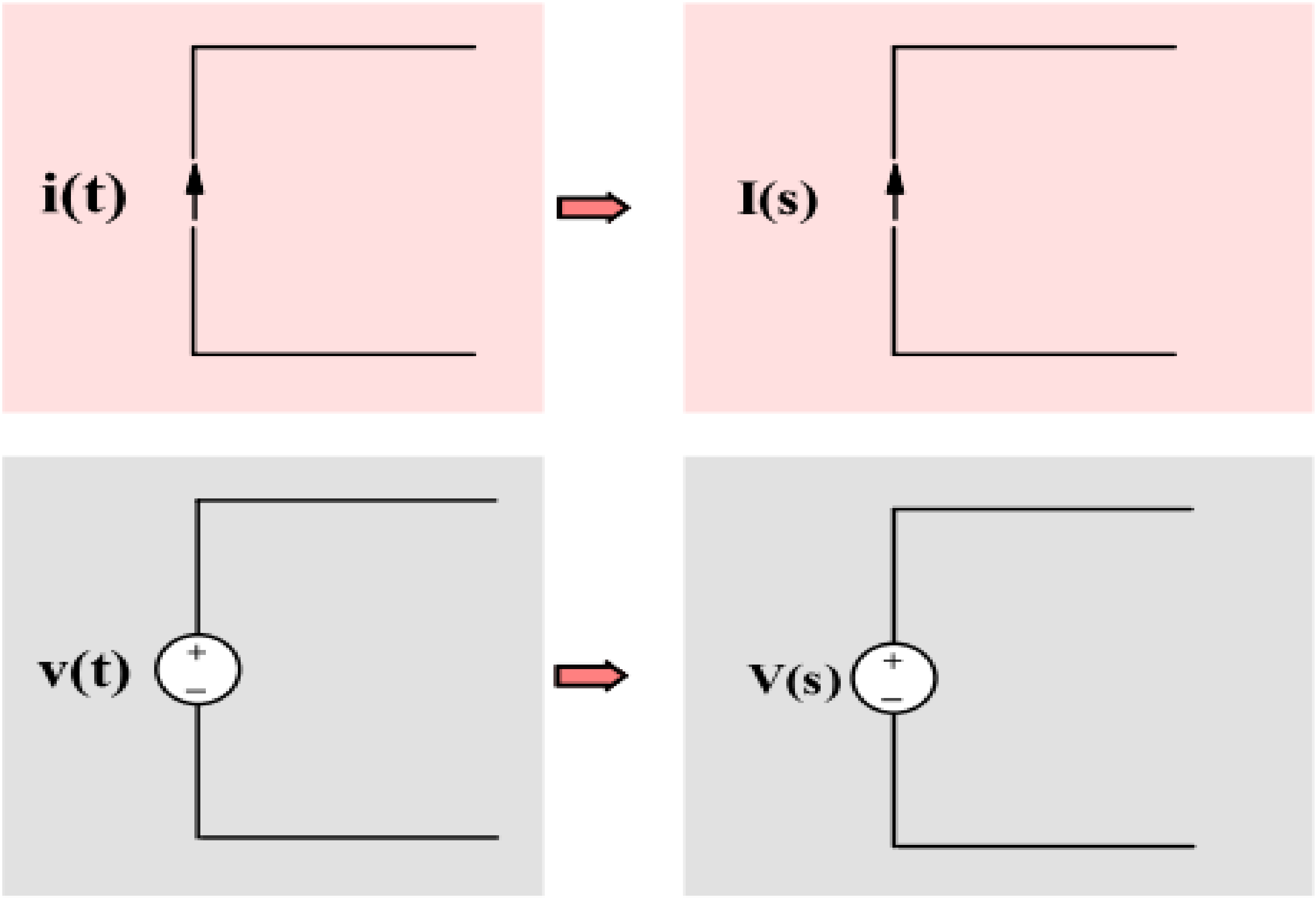
For the complete analysis
Check Chapter 16 Schaum Series (Old version)

Transient Analysis using Laplace Transform

- Laplace transform is considered one of the most important tools in Electrical Engineering
- It can be used for:

  - ✓ Solving differential equations
  - ✓ Circuit analysis (Transient and general circuit analysis)
  - ✓ Digital Signal processing in Communications and
  - ✓ Digital Control

Circuit Elements in the "S" Domain



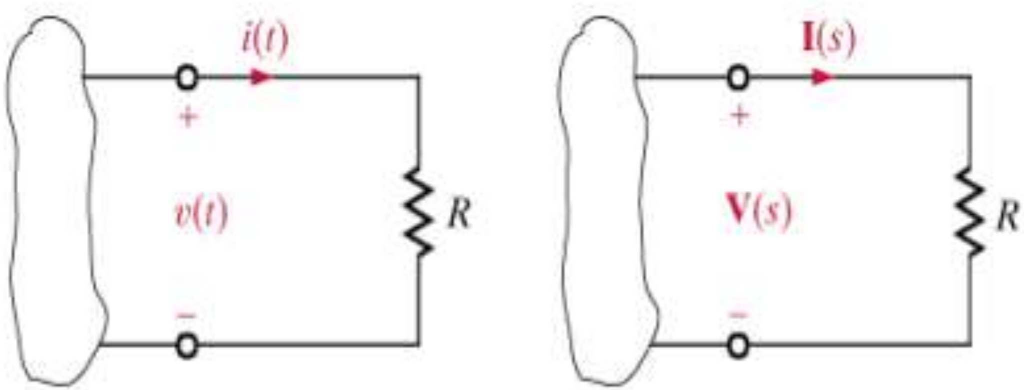
Circuit Elements in the "S" Domain

Circuit Element Modeling

The method used so far follows the steps:
1. Write the differential equation model
2. Use Laplace transform to convert the model to an algebraic form

1.0 Resistance

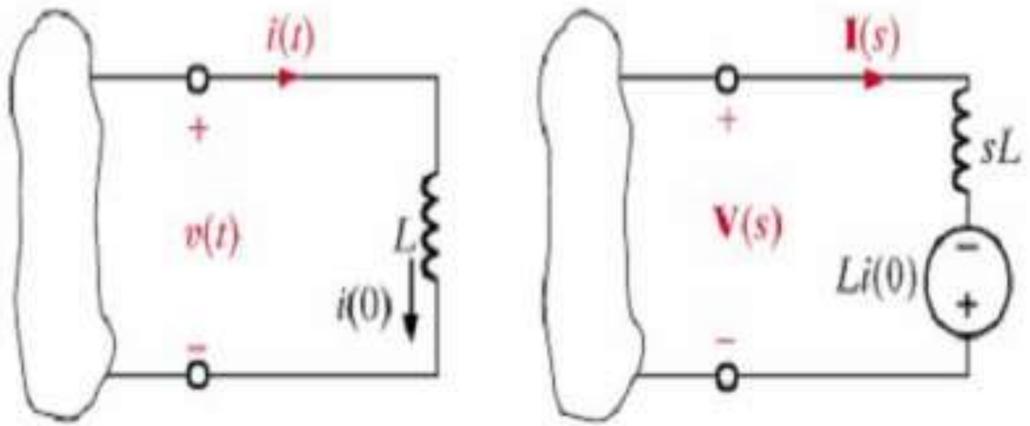
Resistor



$$v(t) = Ri(t) \Rightarrow V(s) = RI(s)$$

Circuit Elements in the "S" Domain

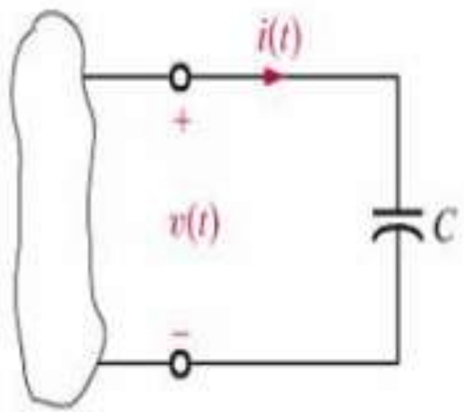
2.0 Inductor

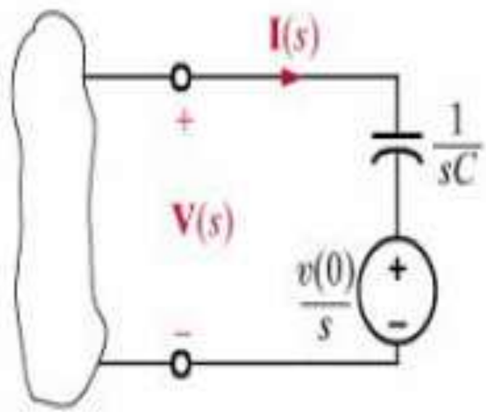


$$v(t) = L\frac{di}{dt}(t) \quad \Rightarrow \quad V(s) = LsI(s) - Li(0)$$

Circuit Elements in the "S" Domain

3.0 Capacitor





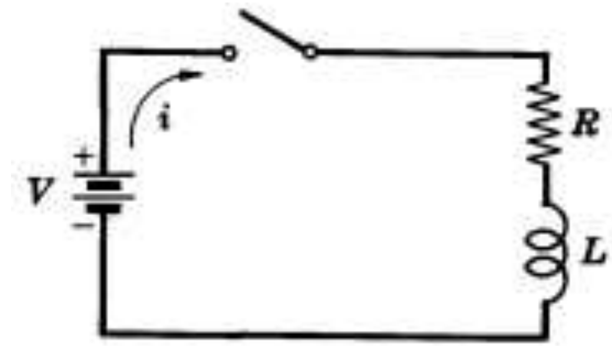
$$v_C(t) = \frac{1}{C}\int_0^t i(t)\,dt + v_C(0)$$

$$V(s) = \frac{1}{Cs} I(s) + \frac{v(0)}{s}$$

First-Order RL Transient (Step-Response)

- The switch "S" is closed at t = 0 to allow the step voltage to excite the circuit
- Apply KVL to the circuit in figure:

$$Ri + L\frac{di}{dt} = V$$



- Apply Laplace Transform on both sides

$$R.I(s) + L[s.I(s) - i(0)] = \frac{V}{s}$$

i(0) = 0 >> initial value of the current at t = 0

$$I(s).[R + sL] = \frac{V}{s}$$

$$I(s) = \frac{V}{s[R + sL]} = \frac{V/L}{s[s + R/L]}$$

- Apply the inverse Laplace Transform technique to get the expression of the current i(t)

First-Order RL Transient (Step-Response)

- Use the partial fraction technique

$$I(s) = \frac{V/L}{s[s + R/L]} = \frac{A_1}{s} + \frac{A_2}{s + R/L}$$

- Multiply both sides by $s.(s + R/L)$

$$V/L = A_1.(s + R/L) + A_2.s$$

$$\ldots\ldots = (A_1 + A_2).s + A_1.\frac{R}{L}$$

both sides Compare the coefficients

$$A_1 = V/R \qquad A_2 = -V/R$$

OR

$$A_1 = \{s * I(s)\}|_{s=0} = \frac{V}{R}$$

$$A_2 = \{(s + R/L) * I(s)\}|_{s=-R/L} = -\frac{V}{R}$$

- So, the current in s-domain is given by:

$$I(s) = \frac{V}{R}\left(\frac{1}{s} - \frac{1}{s + R/L}\right)$$

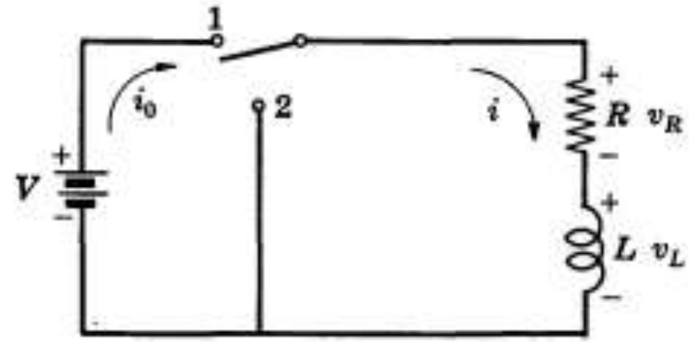
- Apply the inverse Laplace transform :

$$i(t) = \frac{V}{R}(1 - e^{-\frac{R}{L}t}); \ t > 0$$

The same as last lecture

First-Order RL Transient (Discharge)

- The RL circuit shown in Figure contains an initial current of (V/R)
- The Switch "S" is moved to position"2" at t=0



$$L\frac{di}{dt} + Ri = 0$$

- Apply Laplace Transform on both sides

$$R.I(s) + L[s.I(s) - i(0)] = 0$$

i(0) = V/R >> initial value of the current at t = 0

$$R.I(s) + L[s.I(s) - V/R] = 0 \Rightarrow I(s) = \frac{V/R}{[s + R/L]}$$

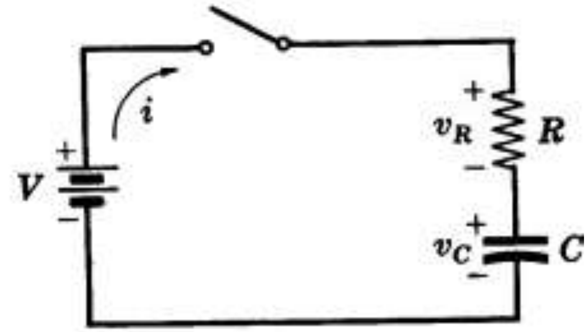
- Apply the inverse Laplace transform :

$$i(t) = \frac{V}{R}.e^{-\frac{R}{L}t} = I_o.e^{-\frac{R}{L}t}; t > 0$$

- The same as before

First-Order RC Transient (Step-Response)

- Assume the switch S is closed at t = 0
- Apply KVL to the series RC circuit shown:



$$[\frac{1}{c}\int i(t).dt + v_c(0)] + R.i(t) = V$$

- Apply Laplace Transform on both sides

$$[\frac{I(s)}{cs} + \frac{v_c(0)}{s}] + R.I(s) = \frac{V}{s}$$

$V_c(0) = 0$ >> initial value of the voltage at t = 0

$$I(s).[R + \frac{1}{cs}] = \frac{V}{s} \qquad I(s) = \frac{V/s}{[R + 1/cs]} = \frac{V/R}{[s + 1/cR]}$$

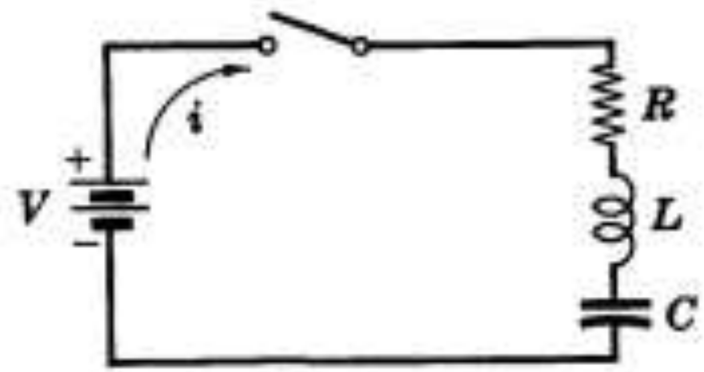
- Apply the inverse Laplace Transform technique to get the expression of the current i(t)

$$i(t) = \frac{V}{R} e^{-\frac{1}{RC}t}; \ t > 0$$

The same as last lecture

Second-Order RLC Transient (Step Response)

- The Switch "S" is closed at t=0
- Applying KVL will produce the following Integro-Differential equation:



$$[\frac{1}{c}\int i(t).dt + v_c(0)] + L\frac{di(t)}{dt} + R.i(t) = V$$

- Apply Laplace Transform on both sides

$$[\frac{I(s)}{cs} + \frac{v_c(0)}{s}] + L.[s.I(s) - i(0)] + R.I(s) = \frac{V}{s}$$

Assume:

$V_c(0) = 0$ & $i(0) = 0$

$$I(s) = \frac{V/s}{[R + sL + 1/cs]} = \frac{V/L}{[s^2 + s.(R/L) + 1/Lc]}$$

- To convert this to time-domain, it will depend on the roots of the denominator which could be expressed as:

(s-S1).(s-S2) >>>>> similar to last lecture (m-m1).(m-m2)

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0$$

Second-Order RLC Transient (Step Response)

$$S_{1,2} = \frac{-R}{2L} \mp \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$S_{1,2} = -\zeta\omega_0 \mp \omega_0\sqrt{\zeta^2 - 1}$$

ζ : expontial dampling ratio

ω_0 : undamped natural frequency

$$S_{1,2} = -\sigma \mp \sqrt{\sigma^2 - \omega_0^{\,2}}$$

- Apply Partial Fraction:

$$I(s) = \frac{V/L}{[s^2 + s.(R/L) + 1/Lc]} = \frac{A}{S - S_1} + \frac{B}{S - S_2}$$

$$A = (S - S_1).I(s)|_{s=s_1} = \frac{V}{2L.\omega_o\sqrt{\zeta^2 - 1}}$$

$$B = (S - S_2).I(s)|_{s=s_2} = \frac{-V}{2L.\omega_o\sqrt{\zeta^2 - 1}} = -A$$

$$I(s) = \frac{V}{2L.\omega_o\sqrt{\zeta^2 - 1}}.\left[\frac{1}{S - S_1} - \frac{1}{S - S_2}\right]$$

Second-Order RLC Transient (Step Response)

- Apply Partial According to the values of the roots , we have 3 scenarios:

1. Over-damped Case

i.e. Two real distinct roots

$$i(t) = \frac{V}{2L.\omega_o\sqrt{\zeta^2 - 1}}.[e^{S_1 t} - e^{S_2 t}]$$

2. Critically-damped Case

i.e. Two real equal roots $\sigma = \omega_o$

$$S_1 = S_2 = -\sigma = \frac{-R}{2L}$$

$$I(s) = \frac{V/L}{[s^2 + s.(R/L) + 1/Lc]} = \frac{V/L}{(S - S_1)^2} = \frac{V/L}{(S + \sigma)^2} = \frac{V/L}{(S + \omega_o)^2}$$

Convert by inverse L.T:

$$\frac{1}{(S+a)^n} \Leftrightarrow \frac{t^{n-1}}{(n-1)!}.e^{-at}$$

$$i(t) = \frac{V}{L}.t.e^{S_1 t} = \frac{V}{L}.t.e^{-\omega_o t}$$

Second-Order RLC Transient (Step Response)

3. Under-damped Case

i.e. Two Complex-conjugate roots

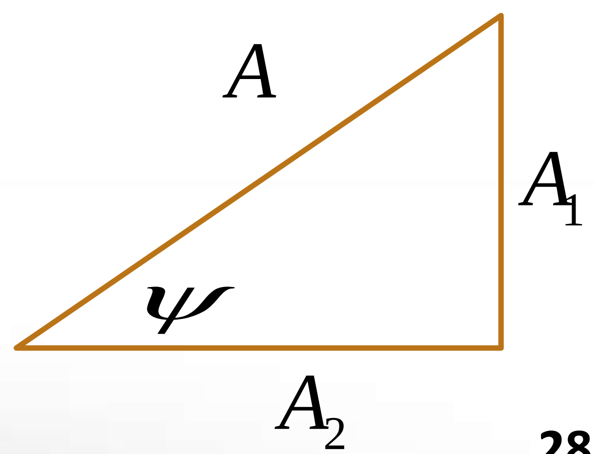
$$S_{1,2} = -\sigma \mp \sqrt{\sigma^2 - \omega_0{}^2} = -\sigma \mp j\omega_d$$

$$i(t) = e^{\sigma.t}[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)]$$

$$\ldots\ldots = Ae^{\sigma.t}[\frac{A_1}{A}\cos(\omega_d t) + \frac{A_2}{A}\sin(\omega_d t)]$$

$$\ldots\ldots = Ae^{\sigma.t}[\sin\psi.\cos(\omega_d t) + \cos\psi.\sin(\omega_d t)]$$

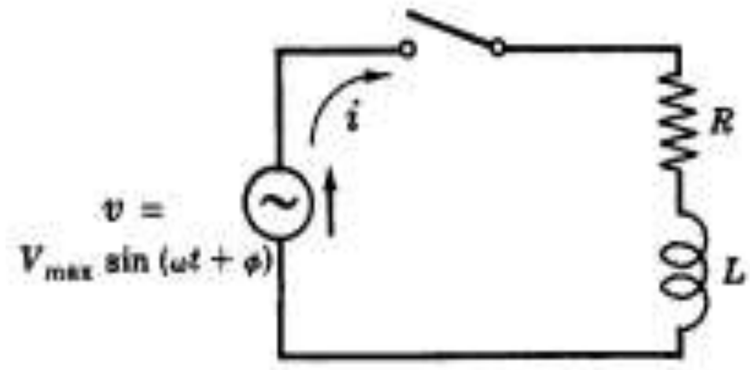
$$\ldots\ldots = Ae^{\sigma.t}\sin(\omega_d t + \psi)$$



RL Sinusoidal Transient

R = 5 ohms, L = 0.01 H, Vm = 100 volts, ϕ = 0, ω = 500

$$Ri + L\frac{di}{dt} = V_{max}\sin(\omega t + \phi)$$



- Apply Laplace Transform on both sides

$$R.I(s) + L[s.I(s) - i(0)] = 100\frac{\omega}{s^2 + \omega^2}$$

i(0) = 0 >> initial value of the current at t = 0

$$I(s).[L(s + \frac{R}{L})] = 100\frac{\omega}{s^2 + \omega^2}$$

$$I(s) = \frac{100.\omega}{L.(s^2 + \omega^2).(s + \frac{R}{L})} = \frac{5x10^6}{(s^2 + \omega^2).(s + 500)}$$

$$S_1 = -500$$
$$S_2 = -j\omega$$
$$S_3 = j\omega$$

RL Sinusoidal Transient

- Use Partial Fraction:

$$I(s) = \frac{5x10^6}{(s^2+\omega^2).(s+500)} = \frac{A_1}{s-j\omega} + \frac{A_2}{s+j\omega} + \frac{A_3}{s+500}$$

OR

$$= \frac{B_1 s + B_2}{s^2+\omega^2} + \frac{A_3}{s+500}$$

Compare to find the constants:

$$I(s) = \frac{10[-s+500]}{s^2+\omega^2} + \frac{10}{s+500} = 10.\frac{500}{s^2+500^2} - 10.\frac{s}{s^2+500^2} + \frac{10}{s+500}$$

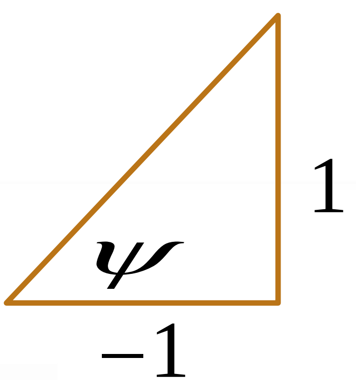
- Use Inverse L.T.

$$i(t) = 10.\sin(500t) - 10.\cos(500t) + 10.e^{-500t}$$

$$i(t) = 10.[\sin(500t) - \cos(500t)] + 10.e^{-500t}$$

$$A = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

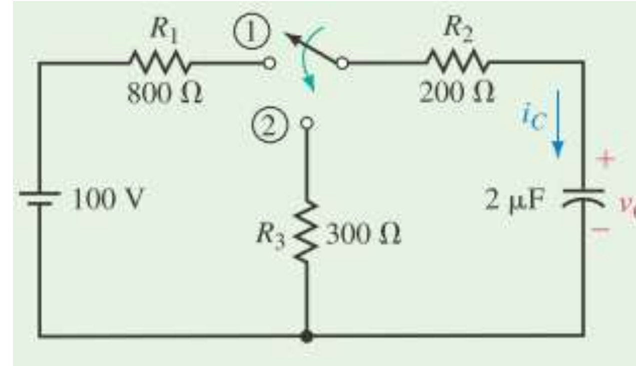
$$\psi = \tan^{-1}(1/-1) = 45^o$$

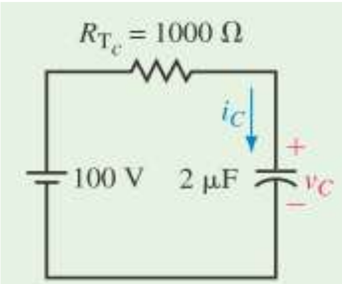


$$i(t) = 10.\sqrt{2}.\sin(500t - 45^o) + 10.e^{-500t} = I_{s.s} + I_{tr}$$

Examples

The capacitor of Figure 11–24(a) is uncharged. The switch is moved to position 1 for 10 ms, then to position 2, where it remains.



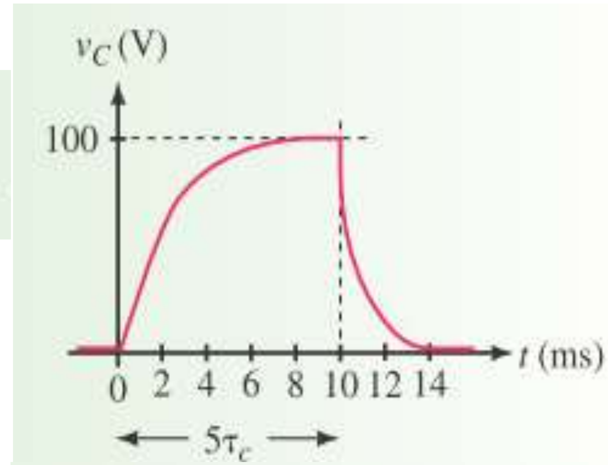


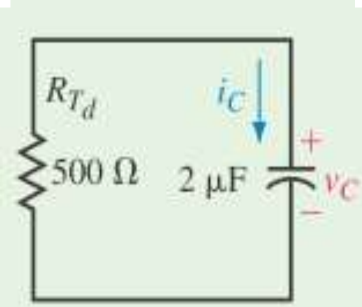
(b) Charging circuit
$R_{T_c} = R_1 + R_2$

$$\tau_C = (R_1 + R_2)C = (1\ \text{k}\Omega)(2\ \mu\text{F}) = 2.0\ \text{ms}.$$

a. $v_C = E(1 - e^{-t/\tau_C}) = 100(1 - e^{-500t})\ \text{V}$

b. $i_C = \dfrac{E}{R_{T_c}} e^{-t/\tau_c} = \dfrac{100}{1000} e^{-500t} = 100e^{-500t}\ \text{mA}$



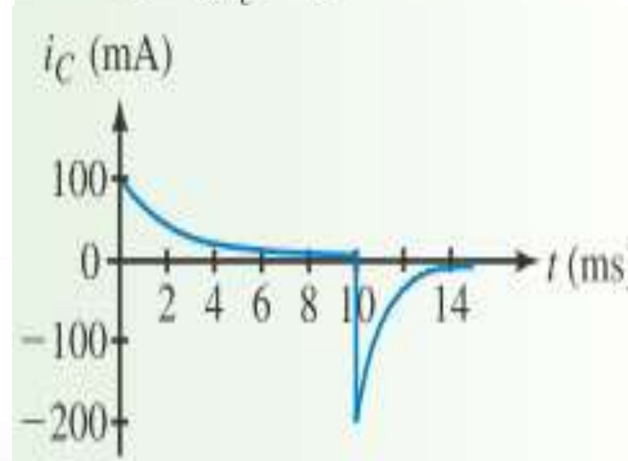


(c) Discharging circuit
$V_0 = 100$ V at $t = 0$ s

$$\tau_d = (500\ \Omega)(2\ \mu\text{F}) = 1.0\ \text{ms}$$

$$v_C = V_0 e^{-t/\tau_d} = 100e^{-1000t}\ \text{V}$$

$$i_C = -\frac{V_0}{R_2 + R_3} e^{-t/\tau_d} = -\frac{100}{500} e^{-1000t} = -200e^{-1000t}\ \text{mA}$$



Note that discharge is more rapid than charge since $\tau_d < \tau_c$.

Check EXAMPLE 11–12 ,13 Miller